Linear Homogeneous equation with constant coefficients

An ordinary linear differential equation of n-th order has of the form

$$\boxed{\frac{d^n y}{dx^n} + P_1 \frac{d^{(n-1)} y}{dx^{(n-1)}} + \cdots + P_{(n-1)} \frac{dy}{dx} + P_n y = X.}$$

where $P_1, P_2, \cdots P_{(n-1)}, P_n$ are constants or functions of x only. Here we discuss about Linear (i,e, degree one) Homogeneous (i,e, R.H.S = 0 i,e, X = 0) equation with constant co-efficients (i,e, $P_1, P_2, \cdots P_{(n-1)}, P_n$ all are constants).

example :-

$$2\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 2y = 0$$ is Linear Homogeneous equation with constant coefficient but $x^2\frac{d^3y}{dx^3} - 2\frac{dy}{dx} = 0$ is not as the coefficient of $\frac{d^3y}{dx^3}$ is not constant here.

Again, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x$ is not Linear Homogenous equation since R.H.S is not zero.

Now let us discuss about the solution of Linear Homogeneous equation with constant co-efficients.

(P.T.O).

Clearly Linear Homogeneous equation with (2
constant coefficients is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{(n-1)}y}{dx^{(n-1)}} + P_2 \frac{d^{(n-2)}y}{dx^{(n-2)}} + \cdots + P_{(n-1)}\frac{dy}{dx} + P_n y = 0 \quad -(1)$$

Let $y = e^{mx}$ be a trial solution of equation (1).

Then $\frac{dy}{dx} = me^{mx}$ $\therefore$ $\frac{d^2y}{dx^2} = m^2 e^{mx}$ and so on.

$\therefore$ we get from (1)

$$\left(m^n + P_1 m^{(n-1)} + P_2 m^{(n-2)} + \cdots + P_{(n-1)} m + P_n\right) e^{mx} = 0$$

Since $e^{mx} \neq 0$ so we get

$$\boxed{m^n + P_1 m^{(n-1)} + P_2 m^{(n-2)} + \cdots + P_n = 0} \quad -(2)$$

This equation is known as auxiliary equation.

[#] Clearly the auxiliary equation has n roots that may be real or complex.

Roots

- Real
  - equal or unequal (distinct)
- Complex
  - equal or unequal (distinct)

We discuss different cases

If roots of the A.E (auxiliary equation) is real and distinct

Let the distinct roots of (2) is $m_1, m_2, \cdots; m_n$ which are real also. Then solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

($c_1, c_2, \cdots c_n$ are arbitrary const)

For example :- i) $3\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 4y = 0$ — (3)

Let $y = e^{mx}$ be a trial solution then we get from (3)

$3m^2 + 8m + 4 = 0$ [Do yourself]

(This is A.E)

we get $m = -2, -2/3$

$\therefore$ solution is $y = c_1 e^{-2x} + c_2 e^{(-2/3)x}$

(Here c_1, c_2 are constants)

ii) $2\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 2y = 0$

Here A.E is $2m^3 - 7m^2 + 7m - 2 = 0$

solving this we get

$(m-1)(m-2)(2m-1) = 0$

$\therefore m = 1, 2, 1/2$.

$\therefore$ solution $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{(\frac{1}{2})x}$

Here c_1, c_2, c_3 are constants.

Next

If roots of the A.E is real and equal

(P.T.O)

If the roots of the A.E is real and equal

i.e, If $m_1 = m_2 = \cdots = m_n = m$ (say)

the solution is of the form

$$y = \left[c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{(n-1)}\right] e^{mx}$$

Here c_i's are constants (where $i = 1, 2, \cdots n$)

<u>For example</u> :- i) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$

A.E $m^2 + 8m + 16 = 0$

$\Rightarrow (m+4)^2 = 0$

$\Rightarrow m = -4, -4$

$\therefore$ soln $y = (c_1 + c_2 x) e^{-4x}$.

ii) <u>Home work</u> :- $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$.

Ans $\rightarrow$ $y = (c_1 + c_2 x + c_3 x^2) e^{-x}$.

<u>Next</u> <u>If roots of the A.E is complex and distinct</u>

It is well known to you that complex and its conjugate complex arise together. So let (n = even number). Therefore complex roots of the form $\alpha_1 \pm i\beta_1$; $\alpha_2 \pm i\beta_2$, $\cdots\cdots$ $\alpha_{(\frac{n}{2})} \pm i\beta_{(\frac{n}{2})}$

Here solution is

$$y = (c_1 \cos\beta_1 x + c_2 \sin\beta_1 x) e^{\alpha_1 x} + (c_3 \cos\beta_2 x + c_4 \sin\beta_2 x) e^{\alpha_2 x}$$
$$+ \cdots + (c_{n-1} \cos\beta_{\frac{n}{2}} x + c_n \sin\beta_{\frac{n}{2}} x) e^{\alpha_{\frac{n}{2}} x}.$$

Examples:-

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 13x = 0$$

A·E $\quad m^2 - 4m + 13 = 0$

$$m = 2 \pm 3i = \alpha \pm i\beta \text{ (say)} \quad (i = \sqrt{-1})$$

so sol^n $\quad y = (c_1 \cos 3x + c_2 \sin 3x)\, e^{2x}$.

$$(c_1, c_2 = \text{const})$$

Note:- i) If roots are $2 \pm 3i$; $5 \pm 4i$ then sol^n will be $\cancel{y = c_1 \cos c}$

$$y = (c_1 \cos 3x + c_2 \sin 3x)\, e^{2x} + (c_3 \cos 4x + c_4 \sin 4x)\, e^{5x}$$

ii) If roots are $\pm i$, $2 \pm 3i$ then

sol^n $\quad y = (c_1 \cos x + c_2 \sin x)\, e^{0 \cdot x} + (c_3 \cos 3x + c_4 \sin 3x)\, e^{2x}$

i,e, $\quad y = (c_1 \cos x + c_2 \sin x) + (c_3 \cos 3x + c_4 \sin 3x)\, e^{2x}$

Next:-

If roots of the A·E is complex and some roots repeats.

If complex roots repeats in p times

i,e, $\quad \alpha \pm i\beta,\ \alpha \pm i\beta,\ \alpha \pm i\beta, \ldots$ p times

then sol^n.

(P.T.O)

$$\cancel{y = c_1 + c_2 x + c_3 x^2 + \cdots + c_p}$$

$$y = e^{\alpha x}\left[(c_1 + c_2 x + c_3 x^2 + \cdots + c_p x^{p-1})\cos\beta x + (c_1 + c_2 x + c_3 x^2 + \cdots + c_p x^{p-1})\sin\beta x\right]$$

For example:-

$$\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 4y = 0$$

A.E $\quad m^4 + 4m^3 + 8m^2 + 8m + 4 = 0.$

$\Rightarrow (m^2 + 2m + 2)^2 = 0$

$\therefore m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{4 - 4\cdot1\cdot2}}{2\cdot1}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$\therefore m = -1 \pm i$

$\therefore$ roots are $\underset{(\alpha \pm i\beta)}{-1 \pm i}, \underset{(\alpha \pm i\beta)}{-1 \pm i} \rightarrow$ equal complex roots.

sol^n

$$y = [(c_1 + c_2 x)\cos x + (c_3 + c_4 \sin x)]e^{-x}.$$

Solve $(D^2+1)^3(D^2+D+1)^2 y = 0$ where $D \equiv \frac{d}{dx}$.

Now A.E $(m^2+1)^3(m^2+m+1)^2 = 0$

$\therefore m^2 + 1 = 0$

$\therefore m^2 = -1 \quad \therefore m = \pm i$

and $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$\therefore$ roots are $\underbrace{\pm i;\ \pm i;\ \pm i}_{\text{thrice times repeat.}};\ \underbrace{-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i;\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}_{\text{(twice times repeats)}}$

solⁿ

$$\therefore y = \left[(c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x\right] e^{0x}$$
$$+ \left[(c_7 \cancel{x} + c_8 x^{\cancel{2}})\cos \frac{\sqrt{3}}{2}x + (c_9 + c_{10} x)\sin \frac{\sqrt{3}}{2}x\right] e^{-\frac{1}{2}x}$$

$$\therefore y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$$
$$+ \left[(c_7 + c_8 x)\cos \frac{\sqrt{3}}{2}x + (c_9 + c_{10} x)\sin \frac{\sqrt{3}}{2}x\right] e^{-\frac{1}{2}x}$$

Note:- 1) For any inquary please contact with me. (~~cont~~ contact no → 8906866150 / 9153561887)
Thank you.

2) Next day, I will discuss Linear non-homogeneous equations. (Basically particular Integrals (P.I))

15. Show that the solution of

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \mu x = 0$$

is $x = e^{-\frac{kt}{2}}(A \cos nt + B \sin nt)$, if $k^2 < 4\mu$ and $n^2 = \mu - \frac{1}{4}k^2$.

16. If $\frac{d^2v}{dx^2} + p^2v = 0$ and $v = v_0$ at $x = x_0$ and $v = 0$ at $x = l$, prove that

$$v = \frac{v_0 \sin p(l-x)}{\sin pl}.$$

17. If $\frac{d^2x}{dt^2} = -n^2x$ and $x = a$, $\frac{dx}{dt} = u$ when $t = 0$, then show that the maximum value of x is $\sqrt{a^2 + \frac{u^2}{n^2}}$. **[B.U. (Hons.) 1985]**

18. Solve $(D^4 - n^4)y = 0$ completely. Now, if $Dy = y = 0$ when $x = 0$ and $x = l$, prove that $y = A(\cos nx - \cosh nx) + B(\sin nx - \sinh nx)$ and $\cos nl \cosh nl = 1$. **[C.U. (Hons.) 1994]**

Answers

1. (a) $y = c_1e^{3x} + c_2e^{-x}$ (b) $x = c_1e^{t} + c_2e^{2t}$
   (c) $y = c_1e^{\frac{1}{2}x} + c_2e^{-\frac{3}{2}x}$ (d) $y = c_1e^{x} + c_2e^{\frac{1}{2}x}$
   (e) $y = c_1e^{ax} + c_2e^{bx}$
2. (a) $y = (c_1 + c_2x)e^{3x}$ (b) $y = (A + Bx)e^{-ax}$
   (c) $y = (c_1 + c_2x)e^{2x}$
3. (a) $y = e^{-4x}(A \cos 3x + B \sin 3x)$ (b) $y = e^{mx}(A \cos nx + B \sin nx)$
4. $y = c_1e^{x} + c_2e^{-2x} + c_3e^{3x}$
5. $y = e^{-x}[(A + Bx) \cos x + (C + Dx) \sin x]$
6. $x = (A_1 + A_2t)e^{t} + A_3e^{2t}$
7. $y = (c_1 + c_2x)e^{2x} + c_3e^{x} + c_4e^{-x}$
8. $y = (c_1 + c_2x + c_3x^2)e^{x} + (c_4 + c_5x)e^{-2x} + c_6e^{2x}$
9. $y = c_1 \cos x + c_2 \sin x + c_3 \cosh x + c_4 \sinh x$
10. $x = Ae^{at} + Be^{-at} + c \cos at + D \sin at$
12. (i) $y = 0$ (ii) $x = 4 \cos 2t + \frac{3}{2} \sin 2t$
14. $s = (s_0 - l) \cos\sqrt{\frac{g}{e}}\, t + l$; $\frac{ds}{dt} = (l - s_0)\sqrt{\frac{g}{e}} \sin\sqrt{\frac{g}{e}}\, t$.

ndent Functions

- **Solve the following equations :**

1. (a) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y = 0$ (b) $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x = 0$

   (c) $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}-3y = 0$ (d) $2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y = 0$

   (e) $\{D^2-(a+b)D+ab\}y = 0$

2. (a) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y = 0$ (b) $(D^2+2mD+m^2)y = 0$

   (c) $y_2-4y_1+4y = 0$

3. (a) $\frac{d^2y}{dx^2}+8\frac{dy}{dx}+25y = 0$ (b) $\frac{d^2y}{dx^2}-2m\frac{dy}{dx}+(m^2+n^2)y = 0$

4. $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y = 0$

5. $\frac{d^4y}{dx^4}+4\frac{d^3y}{dx^3}+8\frac{d^2y}{dx^2}+8\frac{dy}{dx}+4y =$

6. $\frac{d^3y}{dt^3}-4\frac{d^2x}{dt^2}+5\frac{dx}{dt}-4x = 0$

7. $(D^4-4D^3+3D^2+4D-4)y = 0$

8. $(D-1)^3(D^2-4)(D+2)y = 0$

9. $\frac{d^4y}{dx^4}-y = 0$

10. $\frac{d^4x}{dt^4} = a^4x$

11. Show that if $l\frac{d^2\theta}{dt^2}+g\theta = 0$ and if $\theta=\alpha$ and $\frac{d\theta}{dt} = 0$ when $t = 0$, then prove th

    $\theta=\alpha\cos\left(t\sqrt{\frac{g}{l}}\right)$.

12. Solve the equation in the particular cases

    (i) $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y = 0$, given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

    [R.U. (Hons.)

    (ii) $\frac{d^2x}{dt^2}+4x = 0$, when at $t = 0$, $x = 4$ and $\frac{dx}{dt} = 3$.

13. Show that the solution of the equation $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x = 0$, with the relation

    2 and $\frac{dx}{dt}(0) = 0$ will be $x = 2(1+3t)e^{-3t}$.

14. For the equation

$$\frac{d^2s}{dt^2}+\frac{g}{e}(s-l) = 0,$$

l, g, e being constants, find s and $\frac{ds}{dt}$, if $s = s_0$ and $\frac{ds}{dt} = 0$ when $t = 0$

[C.U. (Hons.)